On specification of a class of infinite Jacobi matrices

1992 J. Phys. A: Math. Gen. 25 L403

(http://iopscience.iop.org/0305-4470/25/8/004)

J. Phys. A: Math. Gen. **25** (1992) L403–L408. Printed in the UK

LETTER TO THE EDITOR

On specification of a class of infinite Jacobi matrices

Susumu Yamazaki

Department of Physics, Faculty of Science, Saitama University Urawa, Saitama, 338, Japan

Received 22 October 1991

Abstract. A certain inverse problem concerning the specification of a class of infinite Jacobi matrices in terms of their eigenvalues and eigenfunctions is formulated and solved.

Let L be an infinite Jacobi matrix defined by

$$(L)_{k,k+1} = (L)_{k+1,k} = a_k(0) \qquad (k = 1, 2, \ldots) \tag{1a}$$

$$(L)_{j,k} = 0 \qquad \text{(otherwise)} \tag{1b}$$

where $\{a_k(0) > 0;\ k = 1, 2, \ldots\}$ satisfies

$$a_k(0) \to 0 \qquad (k \to \infty). \tag{2}$$

The condition (2) is equivalent to the one that L stands for a compact (=completely continuous) Hermitian operator on the Hilbert space l_2 [2].

It is known that the eigenvalues of a non-zero compact Hermitian operator on the infinite dimensional separable Hilbert space $\mathscr{H}$ are real, bounded and countably many, with the only possible accumulation point zero. The multiplicity of each non-zero eigenvalue is finite, and the eigenvectors form a complete orthogonal system in $\mathscr{H}$ [2, 5, 6]. In particular, all the eigenvalues of the compact Hermitian operator L are simple. They are infinitely many and appear in positive and negative pairs, and zero is their only accumulation point [8].

Taking account of this, we label the eigenvalues of L as $\{\lambda_{\pm k};\ k = 1, 2, \ldots\}$, where

$$\lambda_{-1} < \lambda_{-2} < \ldots < 0 < \ldots < \lambda_2 < \lambda_1 \tag{3}$$

$$\lambda_{-k} = -\lambda_k \qquad (k = 1, 2, \ldots) \tag{4a}$$

and

$$\lambda_\infty \equiv 0 \qquad \text{(if zero is an eigenvalue)}. \tag{4b}$$

We denote by $\Im$ the set of suffixes of the eigenvalues, i.e.

$$\Im = \{\pm 1, \pm 2, \ldots\} \qquad \text{if zero is not an eigenvalue} \tag{5a}$$

$$\Im = \{\pm 1, \pm 2, \ldots, \infty\} \qquad \text{if zero is an eigenvalue.} \tag{5b}$$

The normalized eigenvector ($\in l_2$) corresponding to the eigenvalue λ_k is denoted by $\psi(\lambda_k) = (\psi_1(\lambda_k), \psi_2(\lambda_k), \ldots)$, so that

$$\|\psi(\lambda_k)\| \equiv \left[\sum_{1 \leqslant l < \infty} |\psi_l(\lambda_k)|^2\right]^{1/2} = 1 \qquad (k \in \Im). \tag{6}$$

0305-4470/92/080403+06\$04.50

From the completeness relation of $\{\psi(\lambda_k);\ k\in\mathfrak{I}\}$, it follows that [8]

$$\sum_{k\in\mathfrak{I}} |\psi_1(\lambda_k)|^2 = 1 \tag{7}$$

where

$$\psi_1(\lambda_k)\neq 0 \qquad (k\in\mathfrak{I}) \tag{8}$$

and

$$|\psi_1(\lambda_k)|^2 = |\psi_1(-\lambda_k)|^2 \qquad (k\in\mathfrak{I}). \tag{9}$$

In terms of these eigenvalues and eigenvectors, $\{a_k(0);\ k=1,2,\ldots\}$ can be expressed as ([8])

$$a_k(0)^2 = P_{k-2}(0)P_{k+1}(0)/P_{k-1}(0)P_k(0) \qquad (k=1,2,\ldots) \tag{10}$$

where $\{P_k(0);\ k=-1,0,1,\ldots\}$ is defined by

$$P_{2l-1}(0) = \det\{(p_{2l-1})_{j,k}\} \qquad (j,k=1,2,\ldots,l;\ l=1,2,\ldots) \tag{11a}$$

$$P_{2l}(0) = \det\{(p_{2l})_{j,k}\} \qquad (j,k=1,2,\ldots,l;\ l=1,2,\ldots) \tag{11b}$$

$$P_0(0) = P_{-1}(0) = 1 \tag{11c}$$

with

$$(p_{2l-1})_{j,k} \equiv \nu_{j+k-2} \qquad (j,k=1,2,\ldots) \tag{12a}$$

$$(p_{2l})_{j,k} \equiv \nu_{j+k-1} \qquad (j,k=1,2,\ldots) \tag{12b}$$

$$\nu_k \equiv \sum_{l\in\mathfrak{I}} \lambda_l^{2k}|\psi_1(\lambda_l)|^2 \qquad (k=0,1,2,\ldots;\ \nu_0=1). \tag{13}$$

Now, the inverse problem of relevance is formulated as follows.

Let $\{\tilde{\lambda}_k;\ k\in\tilde{\mathfrak{I}}\}$ ($\tilde{\mathfrak{I}}$ stands for either (5a) or (5b)) be an infinite sequence of real numbers that satisfies (3) and (4) with circumflexes, and converges to zero. And let $\{\gamma_k;\ k\in\tilde{\mathfrak{I}}\}$ be an infinite sequence of positive numbers satisfying

$$\gamma_k = \gamma_{-k} \qquad (k(\neq\infty)\in\tilde{\mathfrak{I}}) \tag{14}$$

and

$$\sum_{k\in\tilde{\mathfrak{I}}} \gamma_k = 1. \tag{15}$$

In terms of $\{\tilde{\lambda}_k;\ k\in\tilde{\mathfrak{I}}\}$ and $\{\gamma_k;\ k\in\tilde{\mathfrak{I}}\}$, we define $\{a_k(0)>0;\ k=1,2,\ldots\}$ by

$$a_k(0)^2 = \tilde{P}_{k-2}(0)\tilde{P}_{k+1}(0)/\tilde{P}_{k-1}(0)\tilde{P}_k(0) \qquad (k=1,2,\ldots) \tag{16}$$

where $\{\tilde{P}_k(0);\ k=-1,0,1,\ldots\}$ is defined by

$$\tilde{P}_{2l-1}(0) = \det\{(\tilde{p}_{2l-1})_{j,k}\} \qquad (j,k=1,2,\ldots,l;\ l=1,2,\ldots) \tag{17a}$$

$$\tilde{P}_{2l}(0) = \det\{(\tilde{p}_{2l})_{j,k}\} \qquad (j,k=1,2,\ldots,l;\ l=1,2,\ldots) \tag{17b}$$

$$\tilde{P}_0(0) = \tilde{P}_{-1}(0) = 1 \tag{17c}$$

with

$$(\tilde{p}_{2l-1})_{j,k} \equiv \tilde{\nu}_{j+k-2} \qquad (j,k=1,2,\ldots) \tag{18a}$$

$$(\tilde{p}_{2l})_{j,k} \equiv \tilde{\nu}_{j+k-1} \qquad (j,k=1,2,\ldots) \tag{18b}$$

$$\tilde{\nu}_k \equiv \sum_{l\in\tilde{\mathfrak{I}}} \tilde{\lambda}_l^{2k}\gamma_l \qquad (k=0,1,2,\ldots;\ \tilde{\nu}_0=1). \tag{19}$$

We then enquire (i) whether the infinite Jacobi matrix L defined in terms of these $a_k(0)$'s $(k=1,2,\ldots)$ by (1) stands for a compact Hermitian operator on the Hilbert space l_2, i.e. whether it satisfies the condition (2). And if this condition is satisfied, we further ask (ii) whether the prescribed $\{\tilde{\lambda}_k;\ k\in\tilde{\mathfrak{J}}\}$ and $\{\gamma_k;\ k\in\tilde{\mathfrak{J}}\}$ coincide with the eigenvalues and $\{|\psi_1(\lambda_k)|^2;\ k\in\mathfrak{J}\}$ of L, respectively, i.e whether the following equalities hold:

$$\tilde{\mathfrak{J}}=\mathfrak{J} \tag{20}$$

$$\tilde{\lambda}_k=\lambda_k \qquad (k\in\tilde{\mathfrak{J}}(=\mathfrak{J})) \tag{21}$$

$$\gamma_k=|\psi_1(\lambda_k)|^2 \qquad (k\in\tilde{\mathfrak{J}}(=\mathfrak{J})). \tag{22}$$

Remark 1. The $\{P_k(0);\ k=-1,0,1,\ldots\}$ and $\{\tilde{P}_k(0);\ k=-1,0,1,\ldots\}$ defined by (11) and (17), respectively, necessarily become all positive (cf [8]).

The inverse problem formulated here becomes relevant when one treats the semi-infinite system of nonlinear differential equations

$$\dot{a}_k(t)=a_k(t)[a_{k+1}(t)^2-a_{k-1}(t)^2] \qquad (k=1,2,\ldots;\ a_0(t)\equiv 0). \tag{23}$$

For those initial values $\{a_k(0)>0;\ k=1,2,\ldots\}$'s which satisfy the boundary condition (2), an explicit expression of the general solution of (23) has been obtained [8]. Since this solution is expressed by means of $\{\lambda_k;\ k\in\mathfrak{J}\}$ and $\{|\psi_1(\lambda_k)|^2;\ k\in\mathfrak{J}\}$, and since it is practically impossible to know $\{\lambda_k;\ k\in\mathfrak{J}\}$ and $\{|\psi_1(\lambda_k)|^2;\ k\in\mathfrak{J}\}$ by solving the eigenvalue problem associated with the initial Jacobi matrix L, one is obliged to specify *a priori* those $\{\lambda_k;\ k\in\mathfrak{J}\}$ and $\{|\psi_1(\lambda_k)|^2;\ k\in\mathfrak{J}\}$ as the initial value. Such circumstances naturally lead to the above questions.

A similar situation arises when one treats (23) $(a_k(t)>0;\ k=1,2,\ldots)$ under the boundary condition

$$a_k(t)\to 1/2 \qquad (k\to\infty) \tag{24}$$

by an inverse scattering method [3]. In this case, in addition to $\{\lambda_k;\ k\in\mathfrak{J}\}$ and $\{|\psi_1(\lambda_k)|^2;\ k\in\mathfrak{J}\}$, it is necessary to take the phase shift into account. And an inverse problem similar to that formulated above also stems from this.

Now, let us recall that the infinite Jacobi matrix L defined by (1) stands for an operator of Hilbert–Schmidt class on l_2 iff it satisfies

$$\sum_{1\le j,k<\infty}|(L)_{j,k}|^2=2\sum_{1\le k<\infty}a_k(0)^2<\infty. \tag{25}$$

This condition is stronger than (2), and is equivalent to [2, 5]

$$\sum_{k\in\mathfrak{J}}\lambda_k^2<\infty. \tag{26}$$

Corresponding to this, we have the *restricted inverse problem* which replaces, in the above formulation of the problem, the condition

$$\tilde{\lambda}_k\to 0 \qquad (k\to\infty) \tag{27}$$

by

$$\sum_{k\in\tilde{\mathfrak{J}}}\tilde{\lambda}_k^2<\infty. \tag{28}$$

Concerning this restricted problem, we maintain the following:

Proposition. The restricted inverse problem can be solved affirmatively, and L becomes an operator of Hilbert–Schmidt class, i.e. $\{a_k(0);\ k=1,2,\ldots\}$ satisfies (25).

Proof. (i) We notice that $\{a_k(0);\ k=1,2,\ldots\}$ defined by (16) is alternatively expressed as (cf [8])

$$a_k(0)^2=(1/2)[(\dot{\tilde{\mathscr{P}}}_k(0)/\tilde{\mathscr{P}}_k(0))-(\dot{\tilde{\mathscr{P}}}_{k-1}(0)/\tilde{\mathscr{P}}_{k-1}(0))] \qquad (k=1,2,\ldots) \tag{29}$$

with

$$\tilde{\mathscr{P}}_{2l-1}(t)=\sum_{\{k_1,k_2,\ldots,k_l\}}\tilde{R}_{k_1}(0)\tilde{R}_{k_2}(0)\ldots\tilde{R}_{k_l}(0)\Delta(\tilde{\lambda}^2_{k_1},\tilde{\lambda}^2_{k_2},\ldots,\tilde{\lambda}^2_{k_l})^2 \times\exp\{2(\tilde{\lambda}^2_{k_1}+\tilde{\lambda}^2_{k_2}+\ldots+\tilde{\lambda}^2_{k_l})t\} \tag{30a}$$

$$\tilde{\mathscr{P}}_{2l}(t)=\sum_{\{k_1,k_2,\ldots,k_l\}}\tilde{R}_{k_1}(0)\tilde{R}_{k_2}(0)\ldots\tilde{R}_{k_l}(0)\Delta(\tilde{\lambda}^2_{k_1},\tilde{\lambda}^2_{k_2},\ldots,\tilde{\lambda}^2_{k_l})^2 \times\tilde{\lambda}^2_{k_1}\tilde{\lambda}^2_{k_2}\ldots\tilde{\lambda}^2_{k_l}\exp\{2(\tilde{\lambda}^2_{k_1}+\tilde{\lambda}^2_{k_2}+\ldots+\tilde{\lambda}^2_{k_l})t\} \tag{30b}$$

$$\tilde{\mathscr{P}}_0(t)=\tilde{\mathscr{P}}_{-1}(t)\equiv 1 \qquad (-\infty<t<+\infty) \tag{30c}$$

$$\dot{\tilde{\mathscr{P}}}_{2l-1}(t)=\sum_{\{k_1,k_2,\ldots,k_l\}}2(\tilde{\lambda}^2_{k_1}+\tilde{\lambda}^2_{k_2}+\ldots+\tilde{\lambda}^2_{k_l})\tilde{R}_{k_1}(0)\tilde{R}_{k_2}(0)\ldots\tilde{R}_{k_l}(0) \times\Delta(\tilde{\lambda}^2_{k_1},\tilde{\lambda}^2_{k_2},\ldots,\tilde{\lambda}^2_{k_l})^2\exp\{2(\tilde{\lambda}^2_{k_1}+\tilde{\lambda}^2_{k_2}+\ldots+\tilde{\lambda}^2_{k_l})t\} \tag{31a}$$

$$\dot{\tilde{\mathscr{P}}}_{2l}(t)=\sum_{\{k_1,k_2,\ldots,k_l\}}2(\tilde{\lambda}^2_{k_1}+\tilde{\lambda}^2_{k_2}+\ldots+\tilde{\lambda}^2_{k_l})^2\tilde{R}_{k_1}(0)\tilde{R}_{k_2}(0)\ldots\tilde{R}_{k_l}(0) \times\Delta(\tilde{\lambda}^2_{k_1},\tilde{\lambda}^2_{k_2},\ldots,\tilde{\lambda}^2_{k_l})^2\tilde{\lambda}^2_{k_1}\tilde{\lambda}^2_{k_2}\ldots\tilde{\lambda}^2_{k_l} \times\exp\{2(\tilde{\lambda}^2_{k_1}+\tilde{\lambda}^2_{k_2}+\ldots+\tilde{\lambda}^2_{k_l})t\} \tag{31b}$$

where $\Sigma_{\{k_1,k_2,\ldots,k_l\}}$ denotes summation over all combinations $\{k_1,k_2,\ldots,k_l\}$ $(k_i\in\tilde{\mathfrak{J}})$ $(i=1,2,\ldots,l)$, $\tilde{R}_k(0)/\Sigma_{l\in\tilde{\mathfrak{J}}}\tilde{R}_l(0)(\tilde{R}_k(0)>0;\ k\in\tilde{\mathfrak{J}})$ and $\Delta(\tilde{\lambda}^2_{k_1},\tilde{\lambda}^2_{k_2},\ldots,\tilde{\lambda}^2_{k_l})$ are defined by

$$\tilde{R}_k(0)\Big/\sum_{l\in\tilde{\mathfrak{J}}}\tilde{R}_l(0)=\gamma_k \qquad (k\in\tilde{\mathfrak{J}}) \tag{32}$$

and

$$\Delta(\tilde{\lambda}^2_{k_1},\tilde{\lambda}^2_{k_2},\ldots,\tilde{\lambda}^2_{k_l})=\prod_{1\leqslant i<j\leqslant l}(\tilde{\lambda}^2_{k_j}-\tilde{\lambda}^2_{k_i}) \tag{33}$$

respectively, and the positive term series (30) and (31) converge uniformly in wider sense on $-\infty<t<+\infty$.

From (30) and (31), it follows that

$$\dot{\tilde{\mathscr{P}}}_{2l-1}(t)/\tilde{\mathscr{P}}_{2l-1}(t)<2(\tilde{\lambda}^2_1+\lambda^2_2+\ldots+\tilde{\lambda}^2_l) \qquad (l=1,2,\ldots) \tag{34a}$$

$$\dot{\tilde{\mathscr{P}}}_{2l}(t)/\tilde{\mathscr{P}}_{2l}(t)<2(\tilde{\lambda}^2_1+\tilde{\lambda}^2_2+\ldots+\tilde{\lambda}^2_l) \qquad (l=1,2,\ldots) \tag{34b}$$

so that

$$\dot{\tilde{\mathscr{P}}}_k(t)/\mathscr{P}_k(t)<\sum_{l\in\tilde{\mathfrak{J}}}\tilde{\lambda}^2_l(<\infty) \qquad (k=-1,0,1,\ldots). \tag{35}$$

Hence $\{a_k(t)>0;\ k=1,2,\ldots\}$ defined by

$$a_k(t)^2\equiv(1/2)[(\dot{\tilde{\mathscr{P}}}_k(t)/\tilde{\mathscr{P}}_k(t))-(\dot{\tilde{\mathscr{P}}}_{k-1}(t)/\tilde{P}_{k-1}(t))] \qquad (k=1,2,\ldots) \tag{36}$$

satisfies

$$\begin{aligned}\sum_{1\le k<\infty} a_k(t)^2 &= \lim_{n\to\infty}\sum_{1\le k\le n}(1/2)[(\dot{\tilde{\mathscr{P}}}_k(t)/\tilde{\mathscr{P}}_k(t))-(\dot{\tilde{\mathscr{P}}}_{k-1}(t)/\tilde{\mathscr{P}}_{k-1}(t))]\\ &=(1/2)\lim_{n\to\infty}[\dot{\tilde{\mathscr{P}}}_n(t)/\tilde{\mathscr{P}}_n(t)]\\ &<\sum_{l\in\tilde{\mathfrak{I}}}\tilde{\lambda}_l^2\\ &<\infty \qquad (-\infty<t<\infty).\end{aligned} \tag{37}$$

In particular, $\{a_k(0)>0;\ k=1,2,\ldots\}$ satisfies (25).

(ii) The $\{a_k(t)>0;\ k=1,2,\ldots\}$ defined by (36) becomes a solution of (23) (cf [8]) with the prescribed initial value (16) or (29). On account of (37), this solution belongs to l_2.

In terms of $\{\tilde{\mathscr{P}}_k(t);\ k=-1,0,1,\ldots\}$ and arbitrarily specified $x_1(0)$ ($\in\mathbb{R}$), we further define $\{x_k(t);\ k=1,2,\ldots\}$ by

$$x_k(t)\equiv x_1(0)+\ln[\tilde{\mathscr{P}}_1(0)\tilde{\mathscr{P}}_{k-2}(t)/4^{k-1}\tilde{\mathscr{P}}_k(t)] \qquad (k=1,2,\ldots) \tag{38}$$

which becomes a solution of (cf [7])

$$\begin{aligned}&\dot{x}_k(t)=(-1/2)[\exp\{x_{k-1}(t)-x_k(t)\}+\exp\{x_k(t)-x_{k+1}(t)\}]\\ &(k=1,2,\ldots;\ x_0(t)\equiv-\infty).\end{aligned} \tag{39}$$

The $\{a_k(t);\ k=1,2,\ldots\}$ and $\{x_k(t);\ k=1,2,\ldots\}$ thus defined are mutually related by

$$a_k(t)=(1/2)\exp[\{x_k(t)-x_{k+1}(t)\}/2] \qquad (k=1,2,\ldots). \tag{40}$$

It follows from (38) and the known results on the system (39) [7] that the asymptotic time behaviours $x_{2l-1}(t\to+\infty)$ $(l=1,2,\ldots)$ and $x_1(t\to-\infty)$ are given, one the one hand, by

$$x_{2l-1}(t\to+\infty)\sim-2\tilde{\lambda}_l^2t+x_1(0)-\ln[4^{2l-2}\gamma_l]-\sum_{1\le k<l-1}\ln[(\tilde{\lambda}_l^2-\tilde{\lambda}_k^2)^2]$$

$$(l=1,2,\ldots) \tag{41}$$

$$x_1(t\to-\infty)=\begin{cases}x_1(0)-\ln[\gamma_\infty] & \text{if } \tilde{\lambda}_\infty(\equiv 0)\in\tilde{\mathfrak{I}} \qquad (42a)\\ +\infty & \text{if } \tilde{\lambda}_\infty(\equiv 0)\notin\tilde{\mathfrak{I}} \qquad (42b)\end{cases}$$

and on the other, by

$$x_{2l-1}(t\to+\infty)\sim-2\lambda_l^2t+x_1(0)-\ln[4^{2l-2}|\psi_1(\lambda_l)|^2]-\sum_{1\le k\le l-1}\ln[(\lambda_l^2-\lambda_k^2)^2]$$

$$(l=1,2,\ldots) \tag{43}$$

$$x_1(t\to-\infty)=\begin{cases}x_1(0)-\ln[|\psi_1(\lambda_\infty)|^2] & \text{if } \lambda_\infty(\equiv 0)\in\mathfrak{I} \qquad (44a)\\ +\infty & \text{if } \lambda_\infty(\equiv 0)\notin\mathfrak{I}. \qquad (44b)\end{cases}$$

It can be shown that the initial value problem of the semi-infinite system of nonlinear differential equations (23) has a unique solution in l_2 on $-\infty<t<\infty$ [9]. Taking account of this and

$$x_1(t)=x_1(0)-2\int_0^t a_1(s)^2\,\mathrm{d}s \tag{45}$$

which follows from (39), we recognize that the initial value problem of (39) has also a unique solution in the space $\mathfrak{S}$ defined by

$$\mathfrak{S}=\{(x_1(t),x_2(t),\ldots);\ x_k(t)\in C^\infty(\mathbb{R})(k=1,2,\ldots),(a_1(t),a_2(t),\ldots)\in l_2\}$$

where $C^\infty(\mathbb{R})$ denotes the set of C^∞-functions on $\mathbb{R}$ to $\mathbb{R}$, and $a_k(t)$ $(k=1,2,\ldots)$ is given by (40).

Since the two solutions of (39), one of which is given by (38) and the other by (2.4) of [7], for the same initial value $(x_1(0), x_2(0),\ldots)$ such that $(a_1(0), a_2(0),\ldots)\in l_2$, both belong to $\mathfrak{S}$, they are the same. So the asymptotic time behaviours of the former, given by (41) and (42), and those of the latter, given by (43) and (44), must coincide with each other. Hence (20)–(22) follows. □

Remark 2. Of two possible cases $(5a)$ and $(5b)$ of distribution of the eigenvalues of L, it is $(5a)$ that was treated in [7]. The derivation of the asymptotic time behaviours $(42a)$ and $(44a)$ in the case $(5b)$ goes similarly as in $(5a)$.

Remark 3. From the viewpoint of the classical moment problem [1], the statement of the proposition concerning the problem (i) can be described alternatively as follows: Let $\{\tilde{\nu}_k\in\mathbb{R};\ k=0,1,2,\ldots\}$ $(\tilde{\nu}_0=1)$ be an infinite sequence of numbers such that $\{\tilde{P}_k(0);\ k=-1,0,1,\ldots\}$ defined in terms of these $\tilde{\nu}_k$'s by (17) are all positive. Then a necessary and sufficient condition in order that $\tilde{\nu}_k$ may stand for the $2k$th moment $(k=0,1,2,\ldots)$ of a normalized $(\int_{-\infty}^{+\infty} d\rho(\lambda)=1)$ sectionally constant non-decreasing function $\rho(\lambda)$ with jumps γ_l only at $\lambda=\tilde{\lambda}_l$ $(l\in\tilde{\mathfrak{I}})$, where $\tilde{\mathfrak{I}}$, $\{\tilde{\lambda}_l;\ l\in\tilde{\mathfrak{I}}\}$ and $\{\gamma_l>0;\ l\in\tilde{\mathfrak{I}}\}$ satisfy (5), (3), (4), (28), and (14), (15) with circumflexes, respectively, i.e. $\tilde{\nu}_k$ $(k=0,1,2,\ldots)$ may be expressed as (19), is that $\{a_k(0)^2;\ k=1,2,\ldots\}$ defined in terms of the above $\{\tilde{P}_k(0);\ k=-1,0,1,\ldots\}$ by the same expression as (16) satisfies (25). □

We have formulated a certain inverse problem concerning the specification of a class of infinite Jacobi matrices, and solved it in the restricted case. We hope to extend the result to the unrestricted case, and also to those Jacobi matrices which have non-zero diagonal elements.

References

[1] Akhiezer N I 1965 *The Classical Moment Problem and Some Related Questions in Analysis* (Edinburgh: Oliver and Boyd)

[2] Akhiezer N I and Glazman I M 1961 *Theory of Linear Operators in Hilbert Space* (New York: Ungar)

[3] Kac M and van Moerbeke P 1975 On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices *Adv. Math.* **16** 160–9

[4] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220

[5] Kato T 1957 *Topological Analysis; Introduction to Theory and Applications* (Tokyo: Kyo-ritsu) (in Japanese)

[6] Smirnov V I 1957 *Course of Higher Mathematics* vol V (Moscow: National Publisher) (Japanese translation 1962 (Tokyo: Kyo-ritsu), German translation 1962 (Berlin: Deutscher Verlag Wissenschaften))

[7] Yamazaki S 1989 On an exponential type dynamical system: $\dot{x}_k=(-1/2)[\exp(x_{k-1}-x_k)+\exp(x_k-x_{k+1})]$ *J. Phys. A: Math. Gen.* **22** 611–6

[8] Yamazaki S 1990 The semi-infinite system of nonlinear differential equations $\dot{A}_k=2A_k(A_{k+1}-A_{k-1})$; methods of integration and asymptotic time behaviours *Nonlinearity* **3** 653–76

[9] Yamazaki S 1991 to be published